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On the solution of three-dimensional inverse heat conduction in finite media

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Abstract—An accurate and stable analytic approach is developed for solving three-dimensional linear inverse heat conduction problems involving the determination of the surface temperature from the knowledge of the time variation of the temperature at the opposite boundary surface. The least-square technique is utilized to compute the unknown parameters associated with solution. The numerical results show that the current method of analysis: is insensitive to measurement errors; remains stable with measurements involving large number of data points taken with extremely small time steps; and can accommodate, with high degree of accuracy, abrupt changes with time in the unknown surface temperature.

INTRODUCTION

THE INVERSE heat conduction problem has numerous applications in various branches of science and engineering. For instance, when it is difficult to locate a thermocouple at the boundary, or the presence of a thermocouple disturbs the boundary conditions, or the thermocouple is located at an interior point. Specific applications include: the determination of the outer surface conditions during the re-entry of a space vehicle, and the surface conditions at the exhaust of a rocket engine; the motion of a projectile over a gun barrel surface; the sliding of a piston in the combustion chamber; melting and ablation; and freezing or quenching of a material process. The inverse problem is also of frequent occurrence in geophysics.

The application of inverse heat conduction problems can be extended to situations involving the determination of the heat transfer coefficient, the contact conductance, the surface heat flux and the internal energy source.

The difficulties in the analysis of inverse problems arise from the fact that the interior conditions are damped and have time lag compared with the applied surface conditions. This phenomenon is pronounced in the three-dimensional inverse problems; furthermore, customary solution of three-dimensional non-homogeneous problems of finite regions contain triple infinite series expressed in terms of sines or cosines which converge slowly. As a result, less information is available about the surface conditions. Also, in most cases the inverse analysis leads to the solution of a Fredholm type of integral equation, which is known [1] to have no unique and stable solution, unless some explicit functional form is specified for the applied surface condition.

An extensive survey of literature reveals that no work appears to be available on the solution of three-dimensional inverse heat conduction problems.

In this paper we consider a three-dimensional inverse problem and develop a rapidly converging solution over the whole time domain by utilizing the splitting-up procedure [2] combined with the least-square technique. Such a solution remains stable with small time steps and insensitive to measurement errors. Coupled to these features are the capability of accommodating abrupt change with time in the unknown surface temperature.

ANALYSIS

Consider a rectangular parallelepiped, of constant thermal properties, confined to the region, $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $0 \leqslant z \leqslant c$, and initially at a uniform temperature T_∞. Suddenly at time $t = 0$, an unknown temperature variation $f(t)$ is imposed on the boundary surface at $x = 0$, while the boundary at $x = a$ is kept insulated. Other boundary surfaces are assumed to be maintained at a uniform temperature T_∞. Figure 1 depicts the geometry and coordinates. If we measure temperature in excess of T_∞, the mathematical formulation of the problem is given by:

$$\nabla^2\theta(\mathbf{r},t) = \frac{1}{\alpha}\frac{\partial\theta(\mathbf{r},t)}{\partial t} \quad \text{in} \quad \text{region } R, t > 0 \qquad (1a)$$

$$\theta(\mathbf{r},t) = F(t) \quad \text{at} \quad x = 0, t > 0 \qquad (1b)$$

$$\frac{\partial\theta(\mathbf{r},t)}{\partial x} = 0 \quad \text{at} \quad x = a, t > 0 \qquad (1c)$$

NOMENCLATURE

A_0, A_1, A_2	functions defined in equations (6d–e)
a, b, c	sides of the three-dimensional, rectangular parallelepiped region
E	least-square error, equation (14)
$f(t)$	actual unknown surface temperature
$F(t)$	modified unknown surface temperature, $f(t)-T_\infty$, equation (2)
G_0, G_1, G_2, P_0, P_1, P_2, P_3, P_4, P_5	functions defined in the Appendix
q_k	eigenvalues, equation (6h)
$\mathbf{r}$	space variable vector
R	region
R_t	transient function satisfying problem (7)
S_j	steady-state function satisfying problem (5)
t	time variable [h]
T	temperature
T_∞	initial and environment temperature
U	first-stage solution, equation (3)
V	second-stage solution, equation (10)
x, y, z	rectangular coordinates
y_i	experimental data (measured temperature).

Greek symbols

α	thermal diffusivity
α_j, ϕ_j	coefficients defined in equation (2)
$\beta_m, \gamma_n, \lambda, \delta$	eigenvalues defined by equations (8c), (6g, i) and (8d), respectively
τ_1	time at which abrupt change occurs [h]
θ	temperature, $T-T_\infty$
θ_s	surface temperature
η	time variable, $t-\tau_1$ [h]
ψ_{mnk}	eigenfunction, equation (8b)
$\sum_{nk}^{\infty}$	$\equiv \sum_{n=\text{odd}}^{\infty}\sum_{k=\text{odd}}^{\infty}$, equation (6j)
$\sum_{mnk}^{m}$	$\equiv \sum_{m=1}^{\infty}\sum_{n=\text{odd}}^{\infty}\sum_{k=\text{odd}}^{\infty}$, equation (8e).

$$\theta(\mathbf{r}, t) = 0$$

$$\text{at} \quad y = 0, y = b, z = 0, z = c, t > 0 \quad \text{(1d–g)}$$

$$\theta(\mathbf{r}, t) = 0 \quad \text{for} \quad t = 0, \text{in the region } R, \quad \text{(1h)}$$

where $F(t) = f(t) - T_\infty$ is the modified unknown surface temperature.

We wish to calculate the applied surface temperature $F(t)$ from the temperature measurements taken at the insulated boundary. This location being the farthest from the active boundary ($x = 0$), the results obtained under such conditions provide the strictest test for the accuracy and stability of the present method of analysis.

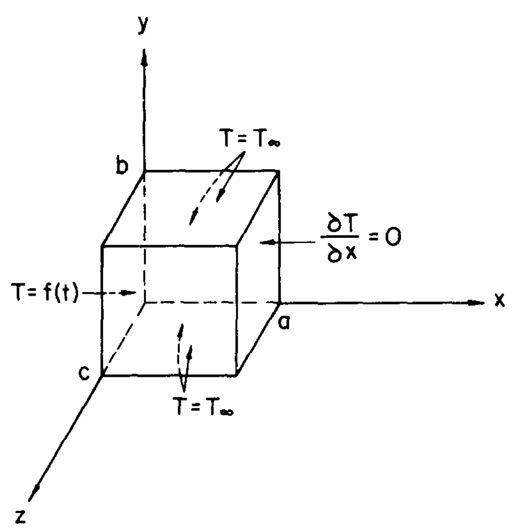

FIG. 1. The geometry and coordinates ($a = c = 1$ ft, $b = \pi$ ft).

Suppose that the applied surface temperature $F(t)$ involves a sharp bend at an unknown τ_1, and represented in the form:

$$F(t) = \begin{cases} \sum_{j=0}^{2} \phi_j \cdot t^j, & t \leqslant \tau_1 \quad \text{(2a)} \\ \sum_{j=0}^{2} \phi_j \cdot \tau_1^j + \sum_{j=0}^{2} \alpha_j \cdot (t-\tau_1)^j, & t > \tau_1 \quad \text{(2b)} \end{cases}$$

where ϕ_js, α_js, and τ_1 are the unknown quantities which are to be determined by the inverse solution. The functional form of the applied temperature as specified by equations (2) includes as special cases most of the variations considered by the previous investigators, and provides a very strict test for the validity of the present method.

Basic to our analysis of the inverse problem is the development of a rapidly converging direct analytic solution over the entire time domain of the problem defined by equations (1), subjected to the unknown surface temperature $F(t)$ which varies with time as specified by equations (2). Such a solution is now developed in two stages: (a) for times $t \leqslant \tau_1$, and (b) for times $t \geqslant \tau_1$ as described below.

(a) $t \leqslant \tau_1$. To distinguish the different stages we set $\theta \equiv U$ for the first stage, and the system (1) takes the form:

$$\nabla^2 U(\mathbf{r}, t) = \frac{1}{\alpha}\frac{\partial U}{\partial t} \quad \text{in region } R, t \leqslant \tau_1 \quad \text{(3a)}$$

$$U(\mathbf{r},t) = \sum_{j=0}^{2} \phi_j t^j \quad \text{at} \quad x = 0, t \leqslant \tau_1 \tag{3b}$$

$$\frac{\partial U}{\partial x} = 0 \quad \text{at} \quad x = a, t \leqslant \tau_1 \tag{3c}$$

$$U(\mathbf{r},t) = 0$$

$$\text{at} \quad y = 0, y = b, z = 0, z = c, t \leqslant \tau_1 \tag{3d–g}$$

$$U(\mathbf{r},t) = 0 \quad \text{for} \quad t = 0, \text{in the region.} \tag{3h}$$

This problem is split up into four simpler problems [2] as

$$U(\mathbf{r},t) = \sum_{j=0}^{2} S_j(\mathbf{r})\cdot t^j + R_t(\mathbf{r},t), \tag{4}$$

where the functions $S_j(\mathbf{r})$s satisfy the following three steady-state problems:

$$\nabla^2 S_j(\mathbf{r}) = \frac{(j+1)}{\alpha} S_{j+1} \quad \text{in region } R \tag{5a}$$

$$S_j(\mathbf{r}) = \phi_j \quad \text{at} \quad x = 0 \tag{5b}$$

$$\frac{\partial S_j}{\partial x} = 0 \quad \text{at} \quad x = a \tag{5c}$$

$$S_j(\mathbf{r}) = 0 \quad \text{at} \quad y = 0, y = b, z = 0, z = c, \tag{5d–g}$$

with $j = 2, 1, 0$ and $S_3 \equiv 0$.

The solution to problems (5) is readily obtained by the integral transform technique [3] as:

$$S_2 = A_0\cdot\phi_0 \tag{6a}$$

$$S_1 = A_0\cdot\phi_1 + 2A_1\cdot\phi_2 \tag{6b}$$

$$S_0 = A_0\cdot\phi_0 + A_1\cdot\phi_1 + A_2\cdot\phi_2, \tag{6c}$$

where:

$$A_0 = \frac{16}{bc}\sum_{nk}^{\infty} \frac{\cosh\lambda(a-x)}{\cosh\lambda a}\cdot\frac{\sin\gamma_n y\cdot\sin q_k\cdot z}{\gamma_n q_k} \tag{6d}$$

$$A_1 = \frac{8}{\alpha bc}\sum_{nk}^{\infty}\left[(a-x)\sinh\lambda(a-x) - \frac{a\sinh\lambda a\cdot\cosh\lambda(a-x)}{\cosh\lambda a}\right] \times \frac{\sin\gamma_n y\cdot\sin q_k z}{\lambda\gamma_n q_k\cosh\lambda a} \tag{6e}$$

$$A_2 = \frac{8}{\alpha^2 bc}\sum_{nk}^{\infty}\left[\left(\frac{a}{\cosh\lambda a}+\frac{1}{2\lambda}\right) \times \frac{a\cdot\sinh\lambda a\cdot\cosh\lambda(a-x)}{\cosh\lambda a} + \frac{(x^2-2ax)}{2}\cosh\lambda(a-x) + \left(\frac{a}{\cosh\lambda a}+\frac{1}{2\lambda}\right)(x-a)\cdot\sinh\lambda(a-x)\right] \times \frac{\sin\gamma_n y\cdot\sin q_k z}{\lambda^2\gamma_n q_k\cosh\lambda a} \tag{6f}$$

$$\gamma_n \equiv \frac{n\pi}{b}, \quad n = 1,3,5,\ldots \tag{6g}$$

$$q_k \equiv \frac{k\pi}{c}, \quad k = 1,3,5,\ldots \tag{6h}$$

$$\gamma_n^2 + q_k^2 \equiv \lambda_{nk}^2 \equiv \lambda^2 \tag{6i}$$

$$\sum_{nk}^{\infty} \equiv \sum_{n=\text{odd}}^{\infty}\sum_{k=\text{odd}}^{\infty}. \tag{6j}$$

The function $R_t(\mathbf{r},t)$ is the solution of the following transient homogeneous problem:

$$\nabla^2 R_t(\mathbf{r},t) = \frac{1}{\alpha}\frac{\partial R_t}{\partial t} \quad \text{in region } R, t \leqslant \tau_1 \tag{7a}$$

$$R_t(\mathbf{r},t) = 0 \quad \text{at} \quad x = 0, t \leqslant \tau_1 \tag{7b}$$

$$\frac{\partial R_t}{\partial x} = 0 \quad \text{at} \quad x = a, t \leqslant \tau_1 \tag{7c}$$

$$R_t(\mathbf{r},t) = 0$$

$$\text{at} \quad y = 0, y = b, z = 0, z = c, t \leqslant \tau_1 \tag{7d–g}$$

$$R_t(\mathbf{r},t) = -S_0 \quad \text{for} \quad t = 0, \text{in the region,} \tag{7h}$$

where S_0 is defined by equation (6c).

The solution to the problem (7) is given by:

$$R_t(\mathbf{r},t) = -\frac{32}{abc}\sum_{mnk}^{\infty}\frac{\beta_m\cdot\psi_{mnk}}{\delta^2\gamma_n q_k} \times \left[\phi_0 - \frac{\phi_1}{\alpha\delta^2} + \frac{2\phi_2}{\alpha^2\delta^4}\right]\cdot e^{-\alpha\delta^2 t}, \tag{8a}$$

where

$$\psi_{mnk} \equiv \sin\beta_m x\cdot\sin\gamma_n y\cdot\sin q_k z \tag{8b}$$

$$\beta_m \equiv \frac{(m-1)\pi}{a}, \quad m = 1,2,3,\ldots \tag{8c}$$

$$\beta_m^2 + \gamma_n^2 + q_k^2 \equiv \delta_{mnk} \equiv \delta^2 \tag{8d}$$

$$\sum_{mnk}^{\infty} \equiv \sum_{m=1}^{\infty}\sum_{n=\text{odd}}^{\infty}\sum_{k=\text{odd}}^{\infty} \tag{8e}$$

γ_n, q_k are defined in equations (6g, h), respectively.

Introducing equations (6) and (8) into (4), the solution for the first-stage problem (3) becomes:

$$U(\mathbf{r},t) = \left[A_0 - \frac{32}{abc}\sum_{mnk}^{\infty}\frac{\beta_m\cdot\psi_{mnk}}{\delta^2\gamma_n q_k}e^{-\alpha\delta^2 t}\right]\phi_0 + \left[A_0\cdot t + A_1 + \frac{32}{\alpha abc}\sum_{mnk}^{\infty} \times \frac{\beta_m\cdot\psi_{mnk}}{\delta^4\gamma_n q_k}e^{-\alpha\delta^2 t}\right]\phi_1 + \left[A_0\cdot t^2 + 2A_1 t + A_2 - \frac{64}{\alpha^2 abc}\sum_{mnk}^{\infty}\frac{\beta_m\cdot\psi_{mnk}}{\delta^6\gamma_n q_k}e^{-\alpha\delta^2 t}\right]\phi_2, \tag{9}$$

where $A_0, A_1, A_2, \gamma_n, q_k, \psi_{mnk}, \beta_m, \delta$, and $\sum_{mnk}^{\infty}$ are defined previously by equations (6d–h) and (8b–e), respectively. The validity of the previous splitting-up procedure defined by equations (5) and (7) can readily be verified by substituting equation (4) into the heat conduction problem (3).

(b) $t > \tau_1$. For convenience in the analysis we let $\eta \equiv t-\tau_1$ and $\theta \equiv V$; then the problem (1) for the second-stage takes the form:

$$\nabla^2 V(\mathbf{r},\eta) = \frac{1}{\alpha}\frac{\partial V}{\partial \eta} \quad \text{in region } R \tag{10a}$$

$$V(\mathbf{r},\eta) = U(0,y,z,\tau_1)+\sum_{j=0}^{2}\alpha_j\eta^j \quad \text{at} \quad x=0, \eta>0 \tag{10b}$$

$$\frac{\partial V}{\partial x} = 0 \quad \text{at} \quad x = b, \eta > 0 \tag{10c}$$

$$V(\mathbf{r},\eta) = 0$$

$$\text{at} \quad y=0, y=b, z=0, z=c, \eta>0 \tag{10d–g}$$

$$V(\mathbf{r},\eta) = U(\mathbf{r},\tau_1) \quad \text{for} \quad \eta = 0, \text{in the region,} \tag{10h}$$

here,

$$U(0,y,z,\tau_1) \equiv \phi_0+\tau_1\phi_1+\tau_2^2\phi_2. \tag{10i}$$

By utilizing the previous splitting-up procedure the solution to the second-stage problem (10) becomes:

$$\begin{aligned}V(\mathbf{r},\eta) = &\left[A_0-\frac{32}{abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^2\gamma_nq_k}e^{-\alpha\delta^2t}\right]\phi_0\\ &+\left[A_0\cdot\tau_1-\frac{32}{\alpha abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^4\gamma_nq_k}e^{-\alpha\delta^2\eta}\right.\\ &\left.+\frac{32}{\alpha abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^4\gamma_nq_k}e^{-\alpha\delta^2t}\right]\phi_1\\ &+\left[A_0\tau_1^2-\frac{64\tau_1}{\alpha abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^4\gamma_nq_k}e^{-\alpha\delta^2\eta}\right.\\ &+\frac{64}{\alpha^2abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^6\gamma_nq_k}e^{-\alpha\delta^2\eta}\\ &\left.-\frac{64}{\alpha^2abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^6\gamma_nq_k}e^{-\alpha\delta^2t}\right]\phi_2\\ &+\left[A_0-\frac{32}{abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^2\gamma_nq_k}e^{-\alpha\delta^2\eta}\right]\alpha_0\\ &+\left[A_1\eta+A_1+\frac{32}{\alpha abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^4\gamma_nq_k}\right.\\ &\left.\times e^{-\alpha\delta^2\eta}\right]\alpha_1+\left[A_1\cdot\eta^2+2A_1\eta\right.\\ &\left.+A_2-\frac{64}{\alpha^2abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^6\gamma_nq_k}e^{-\alpha\delta^2\eta}\right]\alpha_2,\end{aligned} \tag{11}$$

where $A_0, A_1, A_2, \psi_{mnk}, \beta_m, \gamma_n, q_k, \delta$ and $\sum_{mnk}^{\infty}$ are defined previously.

The solution of the first and second stages, equations (9) and (11), can be written, respectively, in more compact form as:

$$U(\mathbf{r},t) = G_0(\mathbf{r},t)\cdot\phi_0+G_1(\mathbf{r},t)\cdot\phi_1 +G_2(\mathbf{r},t)\cdot\phi_2, \quad t\leq\tau_1 \tag{12a}$$

$$\begin{aligned}V(\mathbf{r},\eta) = &P_0(\mathbf{r},t)\cdot\phi_0+P_1(\mathbf{r},\eta,\tau_1)\cdot\phi_1\\ &+P_2(\mathbf{r},\eta,\tau_1)\cdot\phi_2+P_3(\mathbf{r},\eta,\tau_1)\cdot\alpha_0\\ &+P_4(\mathbf{r},\eta,\tau_1)\cdot\alpha_1+P_5(\mathbf{r},\eta,\tau_1)\cdot\alpha_2\\ &+\ldots, t>\tau_1,\end{aligned} \tag{12b}$$

where the various functions appearing in these equations are defined in the Appendix.

Having established the rapidly converging solutions for the first and second stages, the corresponding inverse analysis of the problem defined by equation (1) can be related to these solutions as:

$$\theta(\mathbf{r},t) = \begin{cases} U(\mathbf{r},t), & t\leq\tau_1 \quad (13a)\\ V(\mathbf{r},\eta), & t>\tau_1 \quad (13b)\end{cases}$$

where the ϕs, αs, and τ_1 are the unknown parameters which are to be determined by utilizing experimental readings of the temperature taken at the insulated boundary ($x = a$). Knowing these parameters, the applied surface temperature $F(t)$ is determined from equation (2).

Method of solution

To solve the inverse problem, we now make use of the least-square technique in order to minimize the error involved in the computation of these parameters. Let E be the error between the exact temperature θ_{ei} as computed from equation (13) and the experimental data y_i taken at $x = a$ at times, t_i, $i = 1$ to N. We represent the error E as:

$$E = \sum_{i=1}^{N}(\theta_{ei}-y_i)^2. \tag{14}$$

Note that the solution (12) for the determination of θ_{ei} involves algebraic equations in which the unknown parameter τ_1, representing the time at which the abrupt change of temperature occurs, appears as a multiplier to other unknown coefficients. Therefore, when θ_{ei} is introduced into equation (14), the resulting system becomes a nonlinear least-square equation. Such a nonlinear system is solved in the following manner:

1. The nonlinear least-square equation (14) is separated into two simpler linear problems by choosing an initial small value for τ_1, say $\tau_1^{(1)}$ as:

$$E = \begin{cases}\sum_{i=1}^{M^{(1)}}(U_i-y_i)^2, & t\leq\tau_1^{(1)} \quad (15a)\\ \sum_{i=1}^{N}(V_i-y_i)^2, & t>\tau_1^{(1)} \quad (15b)\end{cases}$$

where,

(a) U and V are defined by equation (12),

(b) $M^{(1)} = \frac{\tau_1^{(1)}}{\Delta t}$ and Δt(time step) $\equiv 0.01$,

(c) $l = M^{(1)}+1$, and N is the total number of readings,
(d) y_i and y_l are the experimental data taken before and after $\tau_1^{(1)}$, respectively.

2. When the least-square error E in equations (15a,b) is minimized with respect to the unknown coefficients, one obtains a 3×3 symmetrical matrix for each of these equations. When these equations are solved for any given value of τ_1 the corresponding coefficients ϕ_js and α_js are determined. The calculations are started with a sufficient small value of τ_1 and carried out with small increments.
3. For each τ_1, the least-square error E is determined from equation (15) by utilizing the coefficients computed in step 2.
4. The calculations are terminated when the value of τ_1 equals the final time of the temperature measurements.
5. The value of τ_1 corresponding to the smallest value of the last-square error E, represents the desired τ_1.

Once the value of τ_1 is established, the coefficients ϕ_j s and α_j s determined in step 2 for this particular value of τ_1. These coefficients represent the desired coefficients for the problem. Knowing the value of τ_1 at which the abrupt change occurs in the surface temperature, and the coefficients ϕ_js and α_js associated with it, the temperature of the front surface is determined from equation (2a) for times $t \leqslant \tau_1$ and from equation (2b) for times $t > \tau_1$ up to the time of the termination of the measurements.

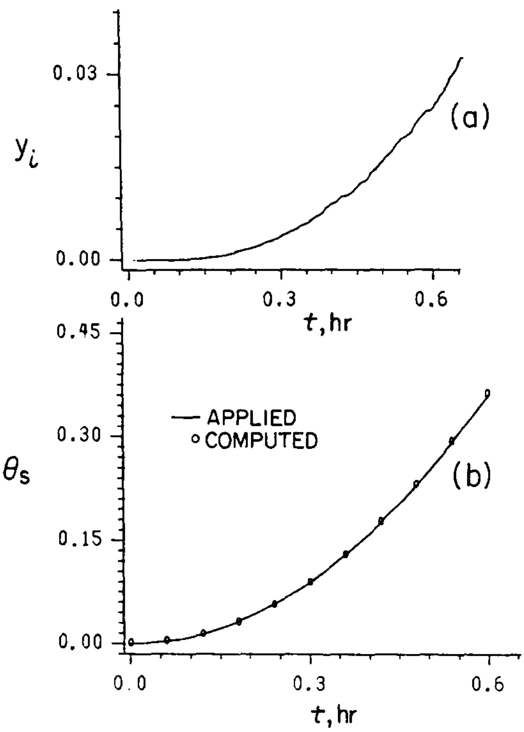

FIG. 2. (a) Measured temperature at the insulated boundary, $x = a$ ($\pm 5\%$ random error); (b) applied and computed surface temperature.

RESULTS AND DISCUSSION

We now present the results of our analysis in predicting the applied surface temperature for situations which, we believe, provide a very strict test condition. Basic steps in the analysis consist of the following:

First we generate the temperature readings from the solution of the appropriate direct problem having an insulated surface at $x = a$ and a specified variation of the surface temperature at $x = 0$, as defined by equations (1). In these calculations we have chosen a thermal diffusivity $\alpha = 0.323 \times 10^{-4}$ m^2 s^{-1} (1.25 ft^2 h^{-1}), $a = c = 0.3048$ m (1 ft) and $b = 0.9575$ m (π ft). The direct temperature readings are taken at small time steps ($\Delta t = 0.01$ h) because small time steps allow more accurate information to be extracted for the variation of the surface conditions. In the present method of analysis, no instability or oscillations were observed with the use of small time steps.

We simulated an *experimental input data* by introducing to the exact measurements, a $\pm 5\%$ normal independent distributed random error with zero mean and a 99% confidence. Using these results as an experimental input data, we then applied the inverse method of analysis described previously in order to determine the applied surface temperature variation.

Figures 2–6 show our predictions of various types of

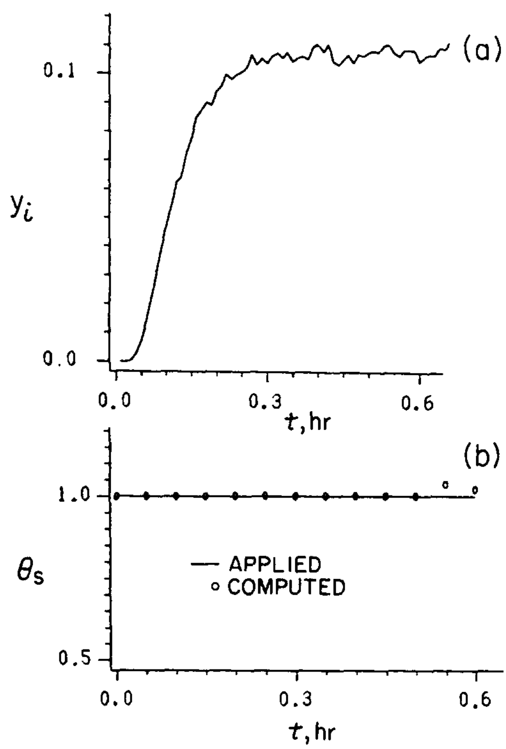

FIG. 3. (a) Measured temperature at the insulated boundary, $x = a$ ($\pm 5\%$ random error); (b) applied and computed surface temperature.

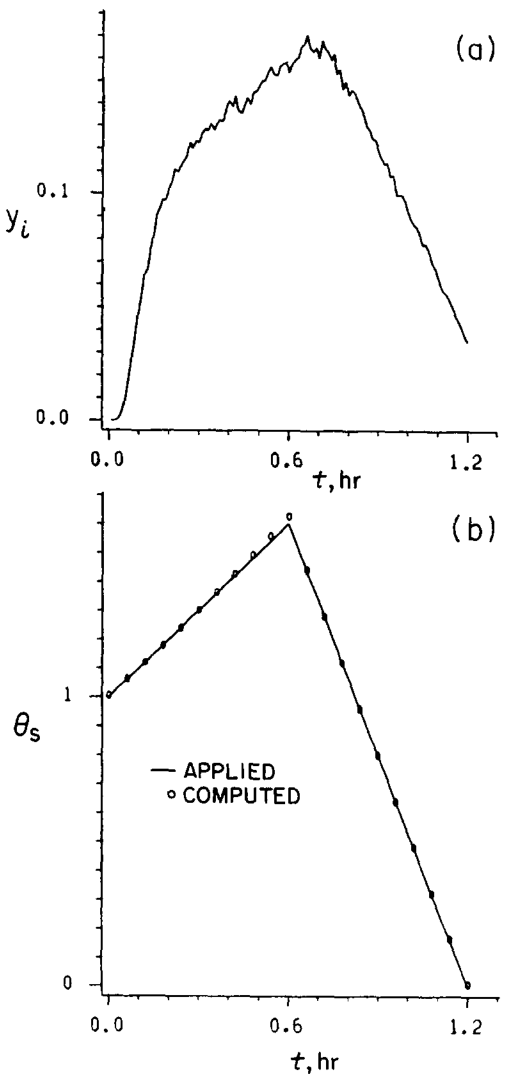

FIG. 4. (a) Measured temperature at the insulated boundary, $x = a(\pm 5\%$ random error); (b) applied and computed surface temperature.

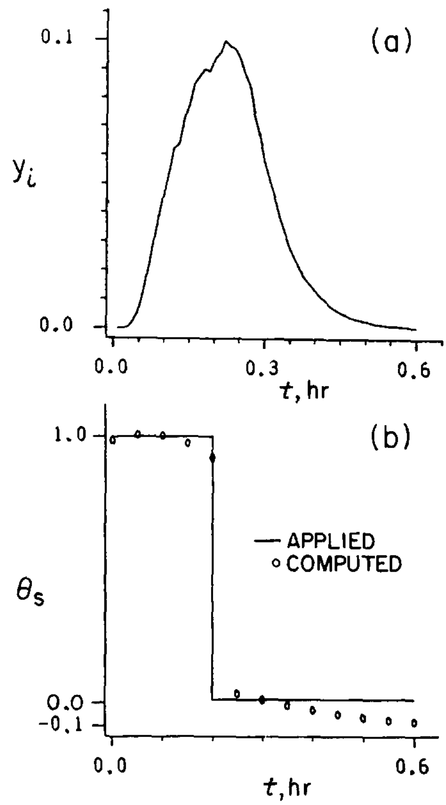

FIG. 5. (a) Measured temperature at the insulated boundary, $x = a(\pm 5\%$ random error); (b) applied and computed surface temperature.

applied surface temperature variations with time. In all these figures, part (a) shows the simulated temperature recordings at the insulated surface $x = a$, while in part (b) we present the exact values of the applied surface temperature with the solid lines and the computed values with circles.

Figures 2 and 3 display, respectively, a parabolic variation and a step change of the applied surface temperature. Clearly, the temperatures predicted by the inverse analysis are in good agreement with the applied surface temperatures for both of these two cases.

Figure 4 shows an applied surface temperature in the form of a chopped-off triangular pulse. Again the prediction with the inverse analysis is good.

Figures 5 and 6 show applied surface temperatures in the form of a rectangular pulse having a short and a long pulse width, respectively. The pulse with a short width poses a difficult case, because the pulse width is small the effects of the time lag on the accuracy of inverse analysis is more pronounced. For such cases, one way to improve the result is to place the thermocouple closer to the surface where the unknown temperature is applied. For the pulse with a wider width as shown in Fig. 6, the prediction with the inverse analysis is good.

Finally, the present analysis clearly demonstrates that the applied surface conditions involving abrupt changes with time can be effectively accommodated with polynomial representations in time over the entire time domain, and the resulting inverse analysis predicts the surface conditions very accurately. All the previous attempts experienced difficulties in developing analytic solutions applicable over the entire time domain when a polynomial representation was used.

CONCLUSIONS

An accurate and stable method of analysis is developed for solving three-dimensional linear inverse heat conduction problems. The present method of solution, which involves a combination of the splitting-up procedure and the least-square technique, has the following features:

1. The solution remains stable for small time steps and

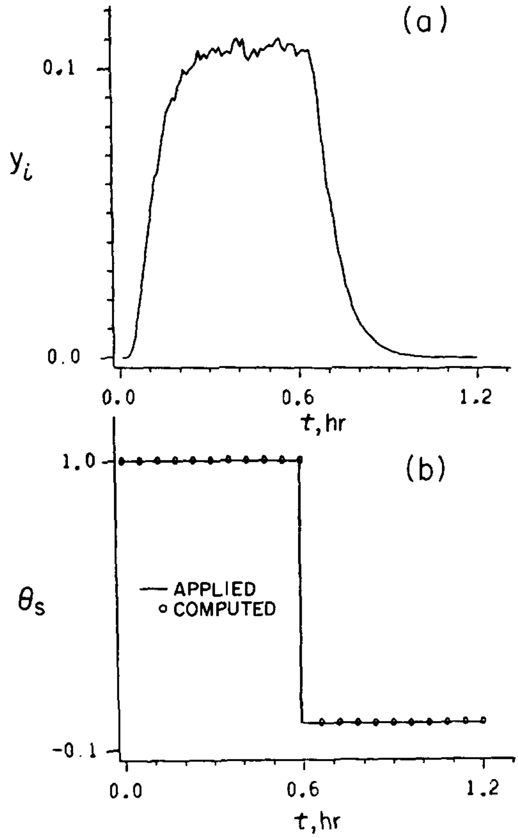

FIG. 6. (a) Measured temperature at the insulated boundary, $x = a(\pm 5\%$ random error); (b) applied and computed surface temperature.

utilizes all the experimental data taken over the whole time domain.
2. The convergence is very fast, because the resulting expressions are in the form of hyperbolic functions.
3. The solution is insensitive to measurement errors.
4. Sharp bends in the surface temperature can be accommodated accurately.
5. The computed results are exact with exact input data, for all cases.
6. The analysis is simple, the inverse solution is easy to program, and computations are efficient.

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APPENDIX

Definition of the functions appearing in equation (12):

$$G_0(\mathbf{r},t) \equiv A_0 - \frac{32}{abc}\sum_{mnk}^{\infty}\frac{\beta_m\cdot\psi_{mnk}}{\delta^2\gamma_n q_k}\,e^{-\alpha\delta^2 t} \quad \text{(A1)}$$

$$G_1(\mathbf{r},t) \equiv A_0\cdot t + A_1 + \frac{32}{\alpha abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^4\gamma_n q_k}\,e^{-\alpha\delta^2 t} \quad \text{(A2)}$$

$$G_3(\mathbf{r},t) \equiv A_0\cdot t^2 + A_1\cdot t + A_3 - \frac{64}{\alpha^2 abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^6\gamma_n q_k} \quad \text{(A3)}$$

$$P_0(\mathbf{r},t) \equiv G_0(\mathbf{r},t) \quad \text{(A4)}$$

$$P_1(\mathbf{r},\eta,\tau_1) \equiv A_0\cdot\tau_1 - \frac{32}{\alpha abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^4\gamma_n q_k}\,e^{-\alpha\delta^2\eta} + \frac{32}{\alpha abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^4\gamma_n q_k}\,e^{-\alpha\delta^2 t} \quad \text{(A5)}$$

$$P_2(\mathbf{r},\eta,\tau_1) \equiv A_0\cdot\tau_1^2 - \frac{64\tau_1}{\alpha abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^4\gamma_n q_k}\,e^{-\alpha\delta^2\eta} + \frac{64}{\alpha^2 abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^6\gamma_n q_k}\cdot e^{-\alpha\delta^2\eta} - \frac{64}{\alpha^2 abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^6\gamma_n q_k}\,e^{-\alpha\delta^2 t} \quad \text{(A6)}$$

$$P_3(\mathbf{r},\eta,\tau_1) \equiv A_0 - \frac{32}{abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^2\gamma_n q_k}\,e^{-\alpha\delta^2\eta} \quad \text{(A7)}$$

$$P_4(\mathbf{r},\eta,\tau_1) \equiv A_1\cdot\eta + A_1 + \frac{32}{\alpha abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^4\gamma_n q_k}\,e^{-\alpha\delta^2\eta} \quad \text{(A8)}$$

$$P_5(\mathbf{r},\eta,\tau_1) \equiv A_1\cdot\eta^2 + 2A_1\cdot\eta + A_2 - \frac{64}{\alpha^2 abc}\sum_{mnk}^{\infty}\frac{\beta_m\psi_{mnk}}{\delta^6\gamma_n q_k}\,e^{-\alpha\delta^2\eta}, \quad \text{(A9)}$$

where A_0, A_1, A_2, β_m, γ_n, q_k, δ, ψ_{mnk}, $\sum_{mnk}^{\infty}$, η are defined in the text.

SUR LA RESOLUTION DU PROBLEME INVERSE DE CONDUCTION THERMIQUE TRIDIMENSIONNELLE DANS UN MILIEU FINI

Résumé—Une approche analytique précise et stable est développée pour résoudre les problèmes linéaires inverses de conduction thermique tridimensionnelle permettant de déterminer la température de surface à partir de la connaissance de la variation dans le temps de la température sur la surface opposée. La technique des moindres carrés est utilisée pour calculer les paramètres inconnus associés à cette solution. Les résultats numériques montrent que la méthode courante d'analyse est insensible aux erreurs de mesure, demeure stable avec des mesures pour un grand nombre de points pris avec un très petit pas de temps, et peut accepter, avec un grand degré de précision, des changements abrupts, dans le temps, de température inconnue de surface.

LÖSUNG DER DREIDIMENSIONALEN INVERSEN WÄRMELEITUNG IN EINEM ENDLICHEN MEDIUM

Zusammenfassung—Es wurde ein genaues und stabiles analytisches Verfahren entwickelt, um das dreidimensionale lineare inverse Wärmeleitproblem, einschließlich der Bestimmung der Oberflächentemperatur bei Kenntnis des zeitlichen Verlaufs der Temperatur der ,gegenüberliegenden Begrenzungsfläche, zu lösen. Die unbekannten Parameter wurden mit Hilfe der Methode der kleinsten Fehlerquadratsumme berechnet. Die numerischen Ergebnisse zeigen, daß die angewandte Rechenmethode unempfindlich hinsichtlich Meßfehlern ist, stabil bleibt bei Messungen, bei denen eine große Zahl von Meßdaten mit extrem kleinen Zeitschrittweiten anfällt, mit hoher Genauigkeit plötzliche zeitliche Änderungen der unbekannten Oberflächentemperatur anpassen kann.

О РЕШЕНИИ ТРЕХМЕРНОЙ ОБРАТНОЙ ЗАДАЧИ ТЕПЛОПРОВОДНОСТИ В ОГРАНИЧЕННЫХ СРЕДАХ

Аннотация—Точный и устойчивый аналитический подход развит для решения трехмерных линейных обратных задач теплопроводности, состоящих в определении температуры поверхности из временной зависимости температуры на противоположной границе. Для расчета неизвестных параметров применяется метод наименьших квадратов. Численные результаты показывают, что предлагаемый метод нечувствителен к ошибкам измерений, остается устойчивым для измерений, включающих большое число течек, взятых при очень малых временных шагах, может соответствовать с высокой степенью точности резким изменениям по времени искомой температуры поверхности.